

FIXED BLOCK CONFIGURATION GROUP DIVISIBLE DESIGNS WITH BLOCK SIZE 6

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ABSTRACT. We present constructions and results about GDDs with two groups and block size 6. We study those GDDs in which each block has configuration (s, t) , that is in which each block has exactly s points from one of the two groups and t points from the other. We show the necessary conditions are sufficient for the existence of $GDD(n, 2, 6; \lambda_1, \lambda_2)$ s with fixed block configuration $(3, 3)$. For configuration $(1, 5)$, we give minimal or near-minimal index examples for all group sizes $n \geq 5$ except $n = 10, 15, 160$, or 190 . For configuration $(2, 4)$, we provide constructions for several families of $GDD(n, 2, 6; \lambda_1, \lambda_2)$ s.¹

1. INTRODUCTION

A group divisible design $GDD(n, m, k; \lambda_1, \lambda_2)$ is a collection of k element subsets of a v -set \mathbf{X} called blocks which satisfies the following properties: each point of \mathbf{X} appears in the same number, r , of the b blocks; the $v = nm$ elements of \mathbf{X} are partitioned into m subsets (called groups) of size n each; pairs of points within the same group are called first associates of each other and appear in λ_1 blocks; pairs of points not in the same group are second associates and appear in λ_2 blocks together. If we require that $m = 2$ and each block intersects one group in s points and $t = k - s$ points in the other, we say the design has fixed block configuration (s, t) .

In [2] the authors settled the existence for group divisible designs with block size three and first and second associates, m groups of size n where $m, n \geq 3$. The problem of finding necessary and sufficient conditions for $m = 2$ or $v = 2n$ and block size four was established in [7]. In [8], the necessary conditions are shown to be sufficient for $3 \leq n \leq 8$. New conditions and results were presented in [4] with three groups and block size four, in particular, constructions were given to show that the necessary conditions are sufficient for all GDDs with three groups and group sizes two, three, and five with two exceptions. In [5], Hurd, Mishra and Sarvate gave new results for general fixed block configuration $GDD(n, 2, k; \lambda_1, \lambda_2)$, as well as new necessary and sufficient conditions for $k = 5$ and configuration $(2, 3)$. Hurd and Sarvate in [6] gave similar results for $k = 5$ and configuration $(1, 4)$. Unless otherwise stated, $m = 2$ is assumed from now on.

The purpose of this article is to establish similar results for GDDs with block size six and two groups. In this paper, we consider each possible configuration type: $(3, 3)$, $(2, 4)$ and $(1, 5)$.

TABLE 1. Possible values of n with respect to λ_1, λ_2

(mod 15)	$\lambda_1 \equiv 0 \pmod{5}$	$\lambda_1 \equiv 1 \pmod{5}$	$\lambda_1 \equiv 2 \pmod{5}$	$\lambda_1 \equiv 3 \pmod{5}$	$\lambda_1 \equiv 4 \pmod{5}$
$\lambda_2 \equiv 0$	Any n	$n \equiv 1 \pmod{5}$	$n \equiv 1 \pmod{5}$	$n \equiv 1 \pmod{5}$	$n \equiv 1 \pmod{5}$
$\lambda_2 \equiv 1$	impossible	$n \equiv 3, 8 \pmod{15}$	$n \equiv 9, 14 \pmod{15}$	$n \equiv 2, 12 \pmod{15}$	impossible
$\lambda_2 \equiv 2$	impossible	$n \equiv 12 \pmod{15}$	$n \equiv 3, 8 \pmod{15}$	impossible	$n \equiv 9 \pmod{15}$
$\lambda_2 \equiv 3$	$n \equiv 0 \pmod{5}$	$n \equiv 4 \pmod{5}$	impossible	$n \equiv 3 \pmod{5}$	$n \equiv 2 \pmod{5}$
$\lambda_2 \equiv 4$	$n \equiv 0 \pmod{15}$	impossible	$n \equiv 2, 12 \pmod{15}$	$n \equiv 9, 14 \pmod{15}$	$n \equiv 3, 8 \pmod{15}$
$\lambda_2 \equiv 5$	$n \equiv 0 \pmod{3}$	$n \equiv 6 \pmod{15}$	$n \equiv 6, 11 \pmod{15}$	$n \equiv 6, 11 \pmod{15}$	$n \equiv 6, 11 \pmod{15}$
$\lambda_2 \equiv 6$	$n \equiv 0 \pmod{5}$	$n \equiv 3 \pmod{5}$	$n \equiv 4 \pmod{5}$	$n \equiv 2 \pmod{5}$	impossible
$\lambda_2 \equiv 7$	impossible	$n \equiv 2, 12 \pmod{15}$	$n \equiv 3, 8 \pmod{15}$	impossible	$n \equiv 9, 14 \pmod{15}$
$\lambda_2 \equiv 8$	impossible	$n \equiv 4, 9 \pmod{15}$	impossible	$n \equiv 3, 8 \pmod{15}$	$n \equiv 2, 12 \pmod{15}$
$\lambda_2 \equiv 9$	$n \equiv 0 \pmod{5}$	impossible	$n \equiv 2 \pmod{5}$	$n \equiv 4 \pmod{5}$	$n \equiv 3 \pmod{5}$
$\lambda_2 \equiv 10$	$n \equiv 0 \pmod{3}$	$n \equiv 6, 11 \pmod{15}$	$n \equiv 6, 11 \pmod{15}$	$n \equiv 6 \pmod{15}$	$n \equiv 6, 11 \pmod{15}$
$\lambda_2 \equiv 11$	impossible	$n \equiv 3 \pmod{15}$	$n \equiv 9, 14 \pmod{15}$	$n \equiv 2, 12 \pmod{15}$	impossible
$\lambda_2 \equiv 12$	$n \equiv 0 \pmod{5}$	$n \equiv 2 \pmod{5}$	$n \equiv 3, 13 \pmod{15}$	impossible	$n \equiv 4, 9 \pmod{15}$
$\lambda_2 \equiv 13$	impossible	$n \equiv 9, 14 \pmod{15}$	impossible	$n \equiv 3, 8 \pmod{15}$	$n \equiv 2, 12 \pmod{15}$
$\lambda_2 \equiv 14$	impossible	impossible	$n \equiv 2, 12 \pmod{15}$	$n \equiv 9, 14 \pmod{15}$	$n \equiv 3 \pmod{5}$

1.1. Necessary Conditions. For GDDs with block size six and two groups there are two necessary conditions on the number of blocks b , and the number of blocks a point appears in r .

Theorem 1.1. *The following conditions are necessary for the existence of a $GDD(n, 2, 6; \lambda_1, \lambda_2)$.*

- (1) *The number of blocks is $b = \frac{\lambda_1(n)(n-1) + \lambda_2 n^2}{15}$.*
- (2) *The number of blocks a point appears in is $r = \frac{\lambda_1(n-1) + \lambda_2 n}{5}$.*

Proof. For condition (1), we count the total number of blocks, b . Each block has $\binom{6}{2} = 15$ pairs. Thus the total number of blocks must be divisible by 15. Consider a point v . There are exactly $\lambda_1(n-1)$ pairs containing another point from the same group, and $\lambda_2 n$ pairs with a point from the other group. Thus the total number of pairs is $15b = \lambda_1(n)(n-1) + \lambda_2 n^2$ and the total number of blocks is $b = \frac{\lambda_1(n)(n-1) + \lambda_2 n^2}{15}$. For condition (2), consider a point v . In any block with v there are 5 pairs containing v and thus the total number of blocks containing v must be divisible by 5. Further v appears in a block λ_1 times with every other point in its same group, which is $n-1$ points, and it appears λ_2 times with every point in the other group (n points in the other group). Thus the total number of blocks that v appears in is $r = \frac{\lambda_1(n-1) + \lambda_2 n}{5}$. \square

These two necessary conditions on b and r determine possibilities for the parameter n and the indices λ_1 and λ_2 . Table 1 summarizes this relationship.

There are at least two other necessary conditions:

Theorem 1.2. *Suppose a $GDD(n, 2, 6; \lambda_1, \lambda_2)$ exists. Then:*

- (1) $b \geq \max(2r - \lambda_1, 2r - \lambda_2)$
- (2) $\lambda_2 \leq 2\lambda_1(n-1)/n$

Proof. For condition (1), consider the set of blocks containing the points x and y . There are r blocks containing x and $r - \lambda_i$ blocks which contain y but do not contain x . So there are at least $2r - \lambda_i$ blocks. For condition (2) let b_6 be the number of blocks with all 6 points from one group, b_5 be the number of blocks with 5 points from 1 group, and the remaining point from the other group, b_4 be the number of blocks with 4 points from 1 group, and the remaining 2 points from the other group, and b_3 be the number of blocks with 3 points from each group. Counting the contribution of these blocks towards the number of pairs of points from the same group in the blocks together gives: $15b_6 + 10b_5 + 7b_4 + 6b_3 = 2\lambda_1 \binom{n}{2} = n(n-1)\lambda_1$. Counting the pairs of points from different groups gives $5b_5 + 8b_4 + 9b_3 = n^2\lambda_2$. Thus we have:

$$\begin{aligned} -15b_6 - 5b_5 + b_4 + 3b_3 &= n^2\lambda_2 - n^2\lambda_1 + n\lambda_1 \leq b_4 + 3b_3 \leq 5b = \\ n[\lambda_1(n-1) + \lambda_2 n]/3 \\ \Rightarrow 3n^2\lambda_2 - 3n^2\lambda_1 + 3n\lambda_1 &\leq n^2\lambda_2 + n^2\lambda_1 - n\lambda_1 \\ \Rightarrow 2n^2\lambda_2 &\leq 4n^2\lambda_1 - 4n\lambda_1 \\ \Rightarrow \lambda_2 &\leq \frac{2(n-1)\lambda_1}{n} \end{aligned} \quad \square$$

Condition (2) shows that while $\lambda_2 \geq \lambda_1$ is possible, we always have $\lambda_2 < 2\lambda_1$. We can apply the theorem to assert the following:

Corollary 1.3. *The family $GDD(n, 2, 6; s, 2st)$ does not exist for any integers $s, t > 0$.*

In [6], Hurd, Mishra and Sarvate proved the following two results for GDDs with fixed block configuration. We repeat their results here.

Theorem 1.4 ([6]). *Suppose a $GDD(n, 2, k; \lambda_1, \lambda_2)$ has configuration (s, t) . Then the number of blocks with s points (respectively t) from the first group is equal to the number of blocks with s points (respectively t) from the second group. Consequently, for any s and t , the number of blocks b is necessarily even.*

Theorem 1.5 ([6]). *For any $GDD(n, 2, k; \lambda_1, \lambda_2)$ with configuration (s, t) , the second index is given by $\lambda_2 = \left(\frac{\lambda_1(n-1)}{n} \right) \left(\frac{k(k-1) - 2\beta}{2\beta} \right)$ where $\beta = \binom{s}{2} + \binom{t}{2}$.*

For the remainder of this paper, we will refer to the results in this section as the “necessary” conditions.

2. GDDs WITH CONFIGURATION (3,3)

In this section, we introduce a basic construction for configuration (3,3) GDDs with specific indices and present the minimal indices for any configuration (3,3) $GDD(n, 2, 6; \lambda_1, \lambda_2)$. We begin by providing an example of a configuration (3,3) GDD where $\lambda_1 = 4$ and $\lambda_2 = 5$.

Example 1: $GDD(6, 2, 6; 4, 5)$. Let $A = \{0, 1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, e, f\}$. Then the $b = 20$ blocks are:

$$\begin{aligned} &\{0, 1, 2, a, b, c\}, \{0, 1, 2, d, e, f\}, \{0, 1, 3, a, b, d\}, \{0, 1, 3, c, e, f\}, \{0, 2, 4, a, c, e\}, \\ &\{0, 2, 4, b, d, f\}, \{0, 3, 5, a, d, f\}, \{0, 3, 5, b, c, e\}, \{0, 4, 5, a, e, f\}, \{0, 4, 5, b, c, d\}, \\ &\{1, 2, 5, b, c, f\}, \{1, 2, 5, a, e, d\}, \{1, 3, 4, b, d, e\}, \{1, 3, 4, a, c, e\}, \{1, 4, 5, b, e, f\}, \\ &\{1, 4, 5, a, c, d\}, \{2, 3, 4, c, d, e\}, \{2, 3, 4, a, b, f\}, \{2, 3, 5, c, d, f\}, \{2, 3, 5, a, b, e\} \end{aligned}$$

By applying Theorem 1.5 to configuration (3,3) GDDs, we get the following result.

Corollary 2.1. *For any configuration $(3, 3)$ GDD($n, 2, 6; \lambda_1, \lambda_2$), we have $\lambda_2 = \frac{3\lambda_1(n-1)}{2n}$.*

2.1. A Basic Construction for Configuration $(3, 3)$. A balanced incomplete block design BIBD(v, k, λ) is a pair (V, B) where V is a set of points with cardinality v and B is a collection of b k -subsets of V called blocks such that each element of V is contained in exactly r blocks and any 2-subset of V is contained in exactly λ blocks. If $k = 3$, we may call the design a triple system, and abbreviate TS(v, λ). We use triple systems in the follow construction.

Theorem 2.2. *If there exists a TS(n, λ) with b blocks and repetition number r , then there exists a configuration $(3, 3)$ GDD($n, 2, 6; \lambda b, r^2$). Further if such a GDD exists, then there exists a TS($n, \lambda b$).*

Proof. Suppose there exists a TS(n, λ). Consider two copies of this triple system, TS₁(n, λ) and TS₂(n, λ). Form the complete bipartite graph G with bipartitions G_1 and G_2 where $V(G_1)$ is the set of blocks of TS₁(n, λ) and $V(G_2)$ is the set of blocks of TS₂(n, λ). The blocks of the desired design are the edge set of G . Consider a pair of first associates. They will appear λ times in TS _{i} (n, λ), $i = 1, 2$. Therefore, in the given construction they will appear together exactly λb times, where b is the number of blocks in a TS(n, λ). Now consider a pair of second associates $\{v_1, v_2\}$ where $v_i \in TS_i(n, \lambda)$. Any point appears exactly r times in a TS(n, λ), thus the pair $\{v_1, v_2\}$ is contained in exactly r^2 blocks of this design.

Now suppose a GDD exists with groups G_1 and G_2 . For each block, remove the points contained in G_1 , and then remove G_1 . What remains is a set of blocks of size 3 on G_2 which have the property that any pair of points occurs in exactly λb blocks. Thus it is a TS($n, \lambda b$). \square

The construction given in Theorem 2.2 can easily be generalized to any configuration (k, k) GDD. Thus we have the following corollary.

Corollary 2.3. *If there exists a BIBD(n, k, λ) with b blocks and repetition number r , then there exists a configuration (k, k) GDD($n, 2, 2k; \lambda b, r^2$).*

2.2. Minimal Indices. There exists a TS(7,1), and thus by Theorem 2.2 there exists a GDD(7,2,6;7,9). From Corollary 2.1, $\lambda_2 = \frac{3\lambda_1(6)}{14} = \frac{9\lambda_1}{7}$, so the construction given in Theorem 2.2 gives a design with the minimum possible indices. However, there also exists a TS(9,1) which means that there exists a GDD(9,2,6;12,16) by Theorem 2.2. In this case we have that $\lambda_2 = \frac{3\lambda_1(8)}{18} = \frac{4\lambda_1}{3}$. Here the minimum values for (λ_1, λ_2) are (3,4). So the construction given in Theorem 2.2 does not give a design with the minimum possible indices. In general, Theorem 2.1 says that for any configuration $(3,3)$ GDD, if for some value of n , the minimum possible indices are (λ_1, λ_2) , then any other GDD with that configuration will have the indices $(w\lambda_1, w\lambda_2)$ for some positive integer w . We can find the minimal indices by using Theorem 2.1 and by the equations given in Theorem 1.1. Any configuration $(3,3)$ GDD with indices $(w\lambda_1, w\lambda_2)$ can be obtained by taking w copies of the blocks in the minimal design. Therefore, we focus on constructing configuration $(3,3)$ GDDs with indices (λ_1, λ_2) . We may then say that the necessary conditions are sufficient for the existence of any configuration $(3,3)$ GDD with that n .

Theorem 2.4. *The minimal indices (λ_1, λ_2) for any configuration $(3, 3)$ GDD($n, 2, 6; \lambda_1, \lambda_2$) are summarized in Table 2.*

TABLE 2. Summary of Minimal Indices for Configuration (3, 3)

n	λ_1	λ_2
$n \equiv 0 \pmod 6$	$2n/3$	$(n-1)$
$n \equiv 1 \pmod 6$	n	$3(n-1)/2$
$n \equiv 2 \pmod 6$	$6n$	$9(n-1)$
$n \equiv 3 \pmod 6$	$n/3$	$(n-1)/2$
$n \equiv 4 \pmod 6$	$2n$	$3(n-1)$
$n \equiv 5 \pmod 6$	$3n$	$9(n-1)/2$

Proof. We know that $\lambda_2 = \frac{3\lambda_1(n-1)}{2n}$ from Theorem 2.1. If $n \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 2$, then $n \equiv 3 \pmod 6$. Thus λ_1 is a multiple of $n/3$ and λ_2 is a multiple of $(n-1)/2$. If $n \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 2$, then $n \equiv 0 \pmod 6$, so λ_1 is a multiple of $2n/3$ and λ_2 is a multiple of $(n-1)$. If $n \equiv 1 \pmod 3$ and $n \equiv 1 \pmod 2$, $n \equiv 1 \pmod 6$, implying λ_1 is a multiple of n and λ_2 is a multiple of $3(n-1)/2$. If $n \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 2$, $n \equiv 4 \pmod 6$, and λ_1 is a multiple of $2n$ and λ_2 is a multiple of $3(n-1)$. If $n \equiv 2 \pmod 3$ and $n \equiv 1 \pmod 2$, then $n \equiv 5 \pmod 6$. This implies that λ_1 is a multiple of n and λ_2 is a multiple of $3(n-1)/2$. However, if we take these values to be the minimal indices, these number of blocks given by Theorem 1.1 would not be integer valued. The smallest values for (λ_1, λ_2) that give integer values for b are $(\lambda_1, \lambda_2) = (3n, \frac{9}{2}(n-1))$. Finally consider when $n \equiv 2 \pmod 3$ and $n \equiv 0 \pmod 2$. Then $n \equiv 2 \pmod 6$, which means that λ_1 is a multiple of $2n$ and λ_2 is a multiple of $3(n-1)$. If we take these values to be the minimal indices, these number of blocks given by Theorem 1.1 would not be integer valued so the smallest values for (λ_1, λ_2) that give integer values for b are $(\lambda_1, \lambda_2) = (6n, 9(n-1))$. \square

3. CONSTRUCTING CONFIGURATION (3,3) GDDs

In this section, we give a similar construction to the one given in Theorem 2.2 based on α -resolvable triple systems. We then show that this construction produces designs with minimal indices for all configuration (3,3) GDDs with block size 6 and 2 groups.

A set of blocks in a design is called a parallel class if it partitions the point set. A partition of the blocks of a design into parallel classes is a resolution, and such a design is called resolvable. An α -parallel class in a design is a set of blocks which contain every point of the design exactly α times. A design that can be resolved into α -parallel classes is called α -resolvable. We may abbreviate an α -resolvable design as an α -RBIBD(n, k, λ). If $\alpha = 1$ then we abbreviate RBIBD(n, k, λ).

The necessary conditions for the existence of a α -RBIBD(n, k, λ) were given by Jungnickle, Mullin and Vanstone in [9].

Theorem 3.1 ([9]). *The necessary conditions for the existence of an α -resolvable BIBD(n, k, λ) are*

- (1) $\lambda(n-1) \equiv 0 \pmod{(k-1)\alpha}$
- (2) $\lambda n(n-1) \equiv 0 \pmod{k(k-1)}$
- (3) $\alpha n \equiv 0 \pmod k$

In the same paper, they also showed that these conditions were sufficient when $k = 3$.

Lemma 3.2 ([9]). *The necessary conditions for the existence of an α -resolvable BIBD($n, 3, \lambda$) are sufficient, except for $n = 6, \alpha = 1$ and $\lambda \equiv 2 \pmod{4}$.*

Vasiga, Furino and Ling [10] showed that the necessary conditions are sufficient for $k = 4$.

Lemma 3.3 ([10]). *The necessary conditions for the existence of an α -resolvable BIBD($n, 4, \lambda$) are sufficient, with the exception of $(\alpha, n, \lambda) = (2, 10, 2)$.*

We use α -resolvable designs to obtain the following result.

Lemma 3.4. *Suppose there exists an α -resolvable TS(n, λ) with s α -parallel classes, where each parallel class contains t blocks. Then there exists a configuration $(3, 3)$ GDD($n, 2, 6; \lambda t, \alpha^2 s$).*

Proof. For $i = 1, 2$, let D_i be an α -resolvable TS(n, λ). Resolve the blocks of D_i into α -parallel classes $C_1^i, C_2^i, \dots, C_s^i$. Construct a graph G in the following manner. For $j = 1, 2, \dots, s$, create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of C_j^1 and $V(G_j^2)$ are the blocks of C_j^2 . Let $G = \bigcup_{j=1}^s G_j$. The edge set of G will form the blocks of the desired design.

Consider a pair of first associates. It will appear in exactly λ blocks of D_i . Therefore, in the given construction, it will appear in λt blocks of size 6. Now consider a pair of second associates $\{v_1, v_2\}$ where $v_1 \in D_1$ and $v_2 \in D_2$. Here v_1 will be matched with v_2 exactly α^2 times per α -parallel class, thus $\lambda_2 = \alpha^2 s$. \square

We now consider values of $n \pmod{6}$ and apply Lemma 3.4 in each case to obtain the desired configuration $(3, 3)$ GDD with minimal indices (λ_1, λ_2) .

Theorem 3.5. *The necessary conditions are sufficient for the existence of a configuration $(3, 3)$ GDD $(n, 2, 6; \frac{n}{3}, \frac{n-1}{2})$ when $n \equiv 3 \pmod{6}$.*

Proof. Let $n \equiv 3 \pmod{6}$. Then by Lemma 3.2 there exists a 1-resolvable TS($n, 1$) with $\frac{n-1}{2}$ parallel classes, each containing $\frac{n}{3}$ blocks. By applying the construction in Lemma 3.4 we obtain a GDD with indices $(\lambda_1, \lambda_2) = (\frac{n}{3}, \frac{n-1}{2})$, which are the minimal indices given in Theorem 2.4. \square

Theorem 3.6. *The necessary conditions are sufficient for the existence of GDD $(n, 2, 6; n, \frac{3}{2}(n-1))$ when $n \equiv 1 \pmod{6}$ with configuration $(3, 3)$.*

Proof. Let $n \equiv 1 \pmod{6}$. By Lemma 3.2 there exists a 3-resolvable TS($n, 1$) with $\frac{n-1}{6}$ 3-parallel classes, each containing n blocks. If we apply the construction in Lemma 3.4, we obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (n, \frac{3(n-1)}{2})$. \square

Theorem 3.7. *The necessary conditions are sufficient for the existence of GDD $(n, 2, 6; 6n, 9(n-1))$ when $n \equiv 2 \pmod{6}$ with configuration $(3, 3)$.*

Proof. Let $n \equiv 2 \pmod{6}$. Then by Lemma 3.2 there exists a 3-resolvable TS($n, 6$) with $(n-1)$ 3-parallel classes, each containing n blocks. Applying Lemma 3.4 yields a GDD with minimal indices $(\lambda_1, \lambda_2) = (6n, 9(n-1))$. \square

Theorem 3.8. *The necessary conditions are sufficient for the existence of GDD $(n, 2, 6; 2n, 3(n-1))$ when $n \equiv 4 \pmod{6}$ with configuration $(3, 3)$.*

Proof. Let $n \equiv 4 \pmod{6}$. By Lemma 3.2, there exists a 3-resolvable $\text{TS}(n, 2)$ with $\frac{n-1}{3}$ 3-parallel classes each containing n blocks. We may apply Lemma 3.4 to obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (2n, 3(n-1))$. \square

Theorem 3.9. *The necessary conditions are sufficient for the existence of GDD $(n, 2, 6; 3n, \frac{9}{2}(n-1))$ when $n \equiv 5 \pmod{6}$ with configuration $(3, 3)$.*

Proof. Let $n \equiv 5 \pmod{6}$. Then by Lemma 3.2 there exists a 3-resolvable $\text{TS}(n, 3)$ with $\frac{n-1}{2}$ 3-parallel classes, each containing n blocks. We may apply Lemma 3.4 to obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (3n, \frac{9(n-1)}{2})$. \square

Theorem 3.10. *The necessary conditions are sufficient for the existence of GDD $(n, 2, 6; \frac{2}{3}n, n-1)$ for $n \equiv 0 \pmod{6}$ with configuration $(3, 3)$.*

Proof. Let $n \equiv 0 \pmod{6}$ with $n \geq 12$. Then by Lemma 3.2 there exists a 1-resolvable $\text{TS}(n, 2)$ with $n-1$ parallel classes, each containing $\frac{n}{3}$ blocks. If we apply the construction given in Lemma 3.4 we obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (\frac{2n}{3}, n-1)$. If $n = 6$, we may not use the construction described in Lemma 3.2. However if $n = 6$, the minimal indices $(\lambda_1, \lambda_2) = (4, 5)$ and Example 1 gives a GDD $(6, 2, 6; 4, 5)$. \square

Since we have given a construction for all possible values of $n \pmod{6}$, we may give the following result.

Theorem 3.11. *The necessary conditions are sufficient for the existence of all configuration $(3, 3)$ GDD $(n, 2, 6; \lambda_1, \lambda_2)$ with minimal indices.*

4. GDDs WITH CONFIGURATION $(2, 4)$

In this section we present the minimal indices for any configuration $(2, 4)$ GDD $(n, 2, 6; \lambda_1, \lambda_2)$. By Theorem 1.5 we have the following relation between λ_1 and λ_2 for any configuration $(2, 4)$ GDD.

Theorem 4.1. *For any configuration $(2, 4)$ GDD $(n, 2, 6; \lambda_1, \lambda_2)$ we have $\lambda_2 = \frac{8\lambda_1(n-1)}{7n}$.*

For any configuration $(2, 4)$ GDD if for some value of n , the minimum possible indices are (λ_1, λ_2) , then any other GDD with that configuration will have the indices $(w\lambda_1, w\lambda_2)$ for some positive integer w . We may find the minimum indices by using the equation in Theorem 4.1, the equations in Theorem 1.1, and the condition in Theorem 1.4. As in the case with configuration $(3, 3)$, we focus on constructing GDDs with minimal indices since we may then say the necessary conditions are sufficient for the existence of any configuration $(2, 4)$ GDD with that n .

Theorem 4.2. *The minimal indices (λ_1, λ_2) for any configuration $(2, 4)$ GDD $(n, 2, 6; \lambda_1, \lambda_2)$ are summarized in Table 3.*

Proof. By Theorem 4.1, we know that $\lambda_2 = \frac{8\lambda_1(n-1)}{7n}$. If $n \not\equiv 1 \pmod{7}$ and n is odd, then this implies that $n \equiv 3, 5, 7, 9, 11, 13 \pmod{14}$. Thus λ_1 is a multiple of $7n$ and λ_2 is a multiple of $8(n-1)$. If $n \equiv 1 \pmod{7}$ and n is odd, then $n \equiv 1 \pmod{14}$. In this case, λ_1 must be a multiple of n and λ_2 a multiple of $(8/7)(n-1)$. If $n \not\equiv 1 \pmod{7}$ and $n \equiv 0 \pmod{8}$, we have that $n \equiv 0, 16, 24, 32, 40, 48 \pmod{56}$, so λ_1 is a multiple of $7n/8$ and λ_2 is a multiple of $n-1$. If $n \not\equiv 1 \pmod{7}$ and $n \equiv 2 \pmod{8}$

TABLE 3. Summary of Minimal Indices for Configuration (2, 4)

n	λ_1	λ_2
$n \equiv 0, 16, 24, 32, 40, 48 \pmod{56}$	$7n/8$	$n - 1$
$n \equiv 2, 6, 10, 14, 18, 26, 30, 34, 38, 42, 46, 54 \pmod{56}$	$7n/2$	$4(n - 1)$
$n \equiv 4, 12, 20, 28, 44, 52 \pmod{56}$	$7n/4$	$2(n - 1)$
$n \equiv 8 \pmod{56}$	$n/8$	$(n - 1)/7$
$n \equiv 22, 50 \pmod{56}$	$n/2$	$4(n - 1)/7$
$n \equiv 36 \pmod{56}$	$n/4$	$2(n - 1)/7$
$n \equiv 3, 5, 7, 9, 11, 13, 17, 19, 21, 23, 25, 27, 31, 33, 35, 37, 39, 41, 45, 47, 49, 51, 53, 55 \pmod{56}$	$7n$	$8(n - 1)$
$n \equiv 1, 15, 29, 43 \pmod{56}$	n	$8(n - 1)/7$

then $n \equiv 2, 10, 18, 26, 34, 42 \pmod{56}$ implying λ_1 is a multiple of $7n/2$ and λ_2 is a multiple of $4(n - 1)$. If $n \not\equiv 1 \pmod{7}$ and $n \equiv 4 \pmod{8}$, $n \equiv 4, 12, 20, 28, 44, 52 \pmod{56}$. Then λ_1 is a multiple of $7n/4$ and λ_2 is a multiple of $2(n - 1)$. If $n \not\equiv 1 \pmod{7}$ and $n \equiv 6 \pmod{8}$, $n \equiv 6, 14, 30, 38, 46, 54 \pmod{56}$, then λ_1 is a multiple of $7n/2$ and λ_2 is a multiple of $4(n - 1)$. If $n \equiv 1 \pmod{7}$ and $n \equiv 0 \pmod{8}$, we have that $n \equiv 8 \pmod{56}$. Here, it follows that λ_1 is a multiple of $n/8$ and λ_2 is a multiple of $(n - 1)/7$. If $n \equiv 1 \pmod{7}$ and $n \equiv 2 \pmod{8}$, we have that $n \equiv 50 \pmod{56}$. Here, it follows that λ_1 is a multiple of $n/2$ and λ_2 is a multiple of $4(n - 1)/7$. If $n \equiv 1 \pmod{7}$ and $n \equiv 4 \pmod{8}$, we have that $n \equiv 36 \pmod{56}$. Here, it follows that λ_1 is a multiple of $n/4$ and λ_2 is a multiple of $2(n - 1)/7$. If $n \equiv 1 \pmod{7}$ and $n \equiv 6 \pmod{8}$, $n \equiv 22 \pmod{56}$, and it follows λ_1 is a multiple of $n/2$ and λ_2 is a multiple of $4(n - 1)/7$. \square

5. CONSTRUCTING (2, 4) GDD($n, 2, 6; \lambda_1, \lambda_2$)

We use the Theorem 4.2 and Lemma 3.3 to construct configuration (2, 4) GDDs with minimal indices, when possible. We begin with a general construction.

Lemma 5.1. *If there exists an α -resolvable BIBD($n, 4, \lambda$) with n even and $\lambda = 3\alpha$, then there exists a configuration (2, 4) GDD($n, 2, 6; \frac{n}{2}(\lambda + \frac{\alpha}{2}), 2\alpha(n - 1)$).*

Proof. Let the two groups be $A = \{1, 2, \dots, n\}$, and $A' = \{1', 2', \dots, n'\}$. Let D be an α -resolvable BIBD($n, 4, \lambda$) on the point set of A . Let F be a 1-factorization of K_n on the point set of A' . Resolve the blocks into α parallel classes. There will be $\lambda(n - 1)/3\alpha = (n - 1)$ classes with $(n\alpha)/4$ blocks in each class. Construct a graph G in the following manner. For $j = 1, 2, \dots, (n - 1)$, create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of an α parallel class and $V(G_j^2)$ are a 1-factor of K_n . If we switch A with A' and repeat the construction, we obtain all desired blocks.

Consider a pair of first associates, $\{x, y\} \in A$. It will appear exactly λ times in D . Therefore in the given construction, it will appear $n\lambda/2$ times when in the first part of the construction. This pair will appear an additional $n\alpha/4$ times when the second part of the construction. Thus $\lambda_1 = \frac{n}{2}(\lambda + \frac{\alpha}{2})$. Now consider a pair of second associates $\{x, y'\}$, where $x \in A$ and $y' \in A'$. Here x will appear with y' exactly $\alpha(n - 1)$ times in both parts of the construction, so $\lambda_2 = 2\alpha(n - 1)$. \square

We use the above construction to obtain the following results:

Corollary 5.2. *Let $n \equiv 2, 6, 10, 14, 18, 26, 30, 34, 38, 42, 46, 54 \pmod{56}$. Then the necessary conditions are sufficient for the existence of a configuration $(2, 4)$ GDD($n, 2, 6; \frac{7n}{2}, 4(n-1)$).*

Proof. Let n be assumed as above. By Lemma 3.3, there exists a 2-resolvable BIBD($n, 4, 6$). Apply Lemma 5.1 to obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (\frac{7n}{2}, 4(n-1))$. \square

Corollary 5.3. *Let $n \equiv 4, 12, 20, 28, 44, 52 \pmod{56}$. Then the necessary conditions are sufficient for the existence of a configuration $(2, 4)$ GDD($n, 2, 6; \frac{7n}{4}, 2(n-1)$).*

Proof. Let n be assumed as above. By Lemma 3.3, there exists a resolvable BIBD($n, 4, 3$). So we may apply Lemma 5.1 to obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (\frac{7n}{4}, 2(n-1))$. \square

We define a near-minimal GDD as a GDD which has indices exactly twice the minimal size.

Corollary 5.4. *If $n \equiv 0, 8 \pmod{24}$ then there exists a near minimal configuration $(2, 4)$ GDD($n, 2, 6; \frac{7n}{4}, 2(n-1)$).*

Proof. Let n be assumed as above. By Lemma 3.3, there exists a resolvable BIBD($n, 4, 3$). Apply Lemma 5.1 to obtain a near-minimal GDD with indices $(\frac{7n}{4}, 2(n-1))$. \square

The above construction gives near-minimal GDDs for $n \equiv 0, 8 \pmod{24}$. The next theorem shows that for $n = 8$, the minimal indices can not be obtained.

Theorem 5.5. *There does not exist a configuration $(2, 4)$ GDD($8, 2, 6; 1, 1$).*

Proof. Assume such a design exists with groups A and B . Then it would have 8 blocks and every point would appear 3 times. Consider a point in the design, x and let its first associates be $\{1, 2, 3, 4, 5, 6, 7\}$. Suppose x appears in 3 blocks which intersect A in 4 points, and $x \in A$ in each of these blocks. Then because there are only 7 other points, there must be a repeated pair in one of these blocks. However, we assumed $\lambda_1 = 1$, so this is not possible. Now suppose x appears in 2 blocks which intersect A in 4 points and $x \in A$ in those blocks. Then x must also appear in a block which intersects B in 2 points and $x \in B$. Let the two partial blocks containing $x \in A$ be $\{x, 1, 2, 3\}$ and $\{x, 4, 5, 6\}$. Without loss, assume the last partial block containing x also contains 1, and $1 \in A$. The part of this block which intersects A may not contain $x, 2, 3$, and we cannot repeat pairs, so 1 must be in a partial block with $\{4, 7\}$. However, there is no additional first associate available to complete this block. Finally assume x appears in one block which intersects A in 4 points and $x \in A$. Without loss, we may assume the partial block containing $x \in A$ be $\{x, 1, 2, 3\}$. Then x appears in 2 blocks which intersect B in 2 points, and $x \in B$. One of these blocks must contain the pair $\{x, 1\}$ where $1 \in A$ and the other block must contain the pair $\{x, 2\}$ where $2 \in A$. However, we have no way to cover the pair $\{x, 3\}$ where $x \in A$ and $3 \in B$ or $x \in B$ or $3 \in A$. Thus this design cannot exist. \square

We use a slightly different construction for $n \equiv 16 \pmod{24}$.

Theorem 5.6. *If $n \equiv 16 \pmod{24}$ then the necessary conditions are sufficient for the existence of a configuration $(2, 4)$ GDD($n, 2, 6; 7n/8, (n-1)$).*

Proof. Let $n \equiv 16 \pmod{24}$, and let $A = \{1, 2, \dots, n\}$ and $A' = \{1', 2', \dots, n'\}$ be the point set for the two groups in the desired design. By Theorem 3.3, there exists a RBIBD($n, 4, 1$). Let D be such a design with point set A . Resolve the blocks of D into parallel classes, $C_1, \dots, C_{(n-1)/3}$. There will be $n/4$ blocks in each parallel class. We construct a 1-factorization of K_n on the point set of A' . On each parallel class $C_j, j = 1, 2, \dots, (n-1)/3$, decompose the blocks of C_j into three 1-factors as follows. For each block $\{a, b, c, d\} \in C_j$ we let $\{\{a', b'\}, \{c', d'\}\} \in F_{j,1}$, $\{\{a', c'\}, \{b', d'\}\} \in F_{j,2}$ and $\{\{a', d'\}, \{b', c'\}\} \in F_{j,3}$.

Now construct a graph G in the following manner. For $j = 1, 2, \dots, (n-1)/3$, construct the complete bipartite graph $G_{j,1}$ with bipartitions $G_{j,1}^1$ and $G_{j,1}^2$ where $V(G_{j,1}^1)$ are, without loss, the first $n/8$ blocks of parallel class C_j and $V(G_{j,1}^2)$ are the 1-factor $F_{j,1}$. Also, create the complete bipartite graph $G_{j,2}$ with bipartitions $G_{j,2}^1$ and $G_{j,2}^2$ where $V(G_{j,2}^1)$ are the last, without loss, $n/8$ blocks of parallel class C_j and $V(G_{j,2}^2)$ are the 1-factor $F_{j,2}$. Construct the complete bipartite graph $G_{j,3}$ with bipartitions $G_{j,3}^1$ and $G_{j,3}^2$ where $V(G_{j,3}^1)$ are the first $n/8$ blocks of parallel class C_j and $V(G_{j,3}^2)$ are the edges of $F_{j,3}$ which were obtained from the first $n/8$ blocks of C_j . Finally construct the complete bipartite graph $G_{j,4}$ with bipartitions $G_{j,4}^1$ and $G_{j,4}^2$ where $V(G_{j,4}^1)$ are the last $n/8$ blocks of parallel class C_j and $V(G_{j,4}^2)$ are the edges of $F_{j,3}$ which are obtained from the last $n/8$ blocks of C_j . If we take the union of all these bipartite graphs, then we obtain half of the blocks of size 6 in the GDD.

To obtain the other half, we switch the roles of A and A' in the design and the 1-factorization. We construct a graph H on the vertex set A, A' in a similar manner to G . For $j = 1, 2, \dots, (n-1)/3$, construct the complete bipartite graph $H_{j,1}$ with bipartitions $H_{j,1}^1$ and $H_{j,1}^2$ where $V(H_{j,1}^1)$ are the last $n/8$ blocks of parallel class C_j and $V(H_{j,1}^2)$ are the 1-factor $F_{j,1}$. Also, construct the complete bipartite graph $H_{j,2}$ with bipartitions $H_{j,2}^1$ and $H_{j,2}^2$ where $V(H_{j,2}^1)$ are the first $n/8$ blocks of parallel class C_j and $V(H_{j,2}^2)$ are the 1-factor $F_{j,2}$. Construct the complete bipartite graph $H_{j,3}$ with bipartitions $H_{j,3}^1$ and $H_{j,3}^2$ where $V(H_{j,3}^1)$ are the first $n/8$ blocks of parallel class C_j and $V(H_{j,3}^2)$ are the edges of $F_{j,3}$ which were obtained from the last $n/8$ blocks of C_j . Finally construct the complete bipartite graph $H_{j,4}$ with bipartitions $H_{j,4}^1$ and $H_{j,4}^2$ where $V(H_{j,4}^1)$ are the last $n/8$ blocks of parallel class C_j and $V(H_{j,4}^2)$ are the edges of $F_{j,3}$ which are obtained from the first $n/8$ blocks of C_j . If we take the union of all these bipartite graphs, then we obtain the other half of the blocks of size 6 in the GDD.

Consider a pair of first associates. In the first part of the construction, when $\{x, y\} \in A$ appears in the BIBD, it will appear exactly once. Thus in the construction, it will be in a block of size 6 exactly $n/2 + n/4 = 3n/4$ times. In the second part of the construction when $\{x, y\}$ is in the role of a 1-factor, it will appear $n/8$ times. Thus $\lambda_1 = 7n/8$. Now consider a pair of second associates, $\{x, y'\}$ where $x \in A$ and $y' \in A'$. Without loss, we may assume $\{x, y'\} \in C_j$ for some j . In part one of the construction, there are 4 cases to consider. Each point is either in the first $n/8$ blocks of C_j or in the last $n/8$ blocks of C_j . Let $C_{j,1}$ denote the first $n/8$ blocks of C_j and $C_{j,2}$ denote the last $n/8$ blocks of C_j . Suppose $x \in C_{j,1}$ and $y' \in C_{j,1}$. Then in the construction, $\{x, y'\}$ appears twice. If $x \in C_{j,1}$ and $y' \in C_{j,2}$, then $\{x, y'\}$ appears once. If $x \in C_{j,2}$ and $y' \in C_{j,1}$, then $\{x, y'\}$ appears once and if $x \in C_{j,2}$ and $y' \in C_{j,2}$, then $\{x, y'\}$ appears twice. In the second part

of the construction when we reverse the roles, if $x \in C_{j,1}$ and $y' \in C_{j,1}$, then $\{x, y'\}$ appears once. If $x \in C_{j,1}$ and $y' \in C_{j,2}$, then $\{x, y'\}$ appears twice. If $x \in C_{j,2}$ and $y' \in C_{j,1}$, then $\{x, y'\}$ appears twice, and if $x \in C_{j,2}$ and $y' \in C_{j,2}$, then $\{x, y'\}$ appears once. Thus for each parallel class, each pair $\{x, y'\}$ appears a total of 3 times. Thus each pair of second associates will appear a total of $3(n-1)/3 = n-1$ times in the construction. \square

Theorem 5.7. *Let $n \equiv 3, 5, 7, 9, 11, 13 \pmod{14}$. Then the necessary conditions are sufficient for the existence of a configuration $(2, 4)$ GDD($n, 2, 6; 7n, 8(n-1)$).*

Proof. Let the two groups be $A = \{1, 2, \dots, n\}$ and $A' = \{1', 2', \dots, n'\}$. By Lemma 3.3, there exists a 4-resolvable BIBD($n, 4, 6$). Let D be such a design with point set A . Resolve the blocks of D into 4-parallel classes. There will be $(n-1)/2$ classes with n blocks in each class. Construct a graph G in the following manner. For $j = 1, 2, \dots, (n-1)/2$ create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of a 4-parallel class and $V(G_j^2)$ are the pairs obtained by developing $\{0', j'\} \pmod{n}$. If we switch A with A' and repeat the same construction, we obtain all desired blocks.

Consider a pair of first associates, $\{x, y\} \in A$. It will appear exactly 6 times in D . Therefore, in the given construction, it will appear $6n$ times in the first part of the construction. This pair will appear an additional n times when in the second part. Thus $\lambda_1 = 7n$. Now consider a pair of second associates $\{x, y'\}$ where $x \in A$ and $y' \in A'$. Here x will be matched with y' exactly $4(n-1)$ times, in each part of the construction, and thus $\lambda_2 = 8(n-1)$. \square

If $n \equiv 1, 15, 29, 43 \pmod{56}$, then the above construction gives a GDD with 7 times the minimal indices. However, the following construction gives a configuration $(2, 4)$ GDD($15, 2, 6; 15, 16$) with minimum possible indices.

Theorem 5.8. *The necessary conditions are sufficient for the existence of a configuration $(2, 4)$ GDD($15, 2, 6; 15, 16$).*

Proof. By Lemma 3.3, there exists a RBIBD($16, 4, 1$). It has 5 parallel classes with 4 blocks in each class. Let $X = \{\infty, 0, 1, 2, \dots, 14\}$ be the points in the RBIBD($16, 4, 1$). Because ∞ appears with every other point exactly once, the blocks of the form $\{\infty, x, y, z\}$ form a partition the set $X \setminus \{\infty\}$. Each block is in one of the 5 parallel classes. For each block $\{\infty, x, y, z\}$, form the pairs $\{x, y\}, \{x, z\}, \{y, z\}$. Let the two groups be $A = \{0, 1, \dots, 14\}$ and $A' = \{0', 1', \dots, 14'\}$. For $j = 1, 2, 3, 4, 5$, create the complete bipartite graph G_j with bipartitions G_{j_1} and G_{j_2} where $V(G_{j_1})$ are the blocks of parallel class j except the block containing ∞ , and $V(G_{j_2})$ are the 15 pairs obtained from the blocks containing ∞ . This gives us half of the desired blocks. To get the rest of the blocks repeat the construction with $V(G_{j_1})$ as the 15 pairs and $V(G_{j_2})$ as the blocks of PC_j .

Consider a pair of first associates, $\{x, y\} \in A$. If $\{x, y\}$ was in a block with ∞ in the RBIBD, then it appears exactly 0 times in the first part of the construction and 15 times in the second part. If $\{x, y\}$ was not in a block with ∞ in the RBIBD, then it appears exactly 15 times in the first part and 0 times in the second part. Therefore, each pair of first associates appears $\lambda_1 = 15$ times. Now consider a pair of second associates $\{x, y'\}$ where $x \in A$ and $y' \in A'$. In the first part, x is in 4 of the blocks and y' is in 2 of the blocks, so $\{x, y'\}$ is in 8 blocks. In the second

part, x is in 2 blocks and y' is in 4 blocks, so $\{x, y'\}$ is again in 8 blocks. Thus, $\lambda_2 = 16$. \square

5.1. Summary of Minimality.

TABLE 4. Summary of Constructions and Minimality for Configuration (2, 4)

n	λ_1	λ_2	
$n \equiv 0, 16, 24, 32, 40, 48 \pmod{56}$ and $n \equiv 16 \pmod{24}$	$7n/8$	$n-1$	minimal
$n \equiv 0, 16, 24, 32, 40, 48 \pmod{56}$ and $n \equiv 0, 8 \pmod{24}$	$7n/8$	$n-1$	near minimal
$n \equiv 2, 10, 18, 26, 34, 42, 6, 14, 30, 38, 46, 54 \pmod{56}$	$7n/2$	$4(n-1)$	minimal
$n \equiv 4, 12, 20, 28, 44, 52 \pmod{56}$	$7n/4$	$2(n-1)$	minimal
$n \equiv 8 \pmod{56}$ and $n \equiv 16 \pmod{24}$	$n/8$	$(n-1)/7$	7 times the minimal
$n \equiv 8 \pmod{56}$ and $n \equiv 0, 8 \pmod{24}$	$n/8$	$(n-1)/7$	14 times the minimal
$n \equiv 22, 50 \pmod{56}$	$n/2$	$4(n-1)/7$	7 times the minimal
$n \equiv 36 \pmod{56}$	$n/4$	$2(n-1)/7$	7 times the minimal
$n \equiv 3, 5, 7, 9, 11, 13 \pmod{14}$	$7n$	$8(n-1)$	minimal
$n \equiv 1 \pmod{14}, n \neq 15$	n	$8(n-1)/7$	7 times the minimal
$n = 15$	15	16	minimal

Table 4 summarizes the results given in this section. It shows when the necessary conditions are sufficient for (2,4) GDDs with minimal indices. Further, the table indicates when the results show the necessary conditions are sufficient for configuration (2, 4) GDDs with near minimal, seven times the minimal possible or fourteen times the minimal possible indices.

6. GDDs WITH CONFIGURATION (1,5)

In this section we focus on the minimal indices for configuration (1, 5) $GDD(n, 2, 6; \lambda_1, \lambda_2)$. Hurd and Sarvate gave a construction for configuration (1, k) $GDD(n, 2, k+1; \lambda_1, \lambda_2)$ using a $BIBD(n, k, \Lambda)$ s [6]. We repeat their result here:

Theorem 6.1 ([6]). *The existence of a $BIBD(n, k, \Lambda)$ implies the existence of a configuration (1, k) $GDD(n, 2, k+1; \lambda_1, \lambda_2)$ with $\lambda_1 = \Lambda n$ and $\lambda_2 = 2\Lambda(n-1)/(k-1)$.*

Further, in [3] Hanani showed the existence of some classes of $BIBD(n, 5, \lambda)$. Using his result and Theorem 6.1 we obtain the following (1, 5) configuration $GDD(n, 2, 6; \lambda_1, \lambda_2)$ s summarized in Table 5.

TABLE 5. Existence of $BIBD(n, 5, \lambda)$ and Resulting Configuration (1, 5) GDDs.

BIBD	Existence	Resulting GDD
$(n, 5, 1)$	$n \equiv 1, 5 \pmod{20}$	$GDD(n, 2, 6; n, (n-1)/2)$
$(n, 5, 2)$	$n \equiv 1, 5 \pmod{10}, n \neq 15$	$GDD(n, 2, 6; 2n, n-1)$
$(n, 5, 4)$	$n \equiv 0, 1 \pmod{10}, n \neq 10, 160, 190$	$GDD(n, 2, 6; 4n, 2(n-1))$
$(n, 5, 5)$	$n \equiv 1 \pmod{4}$	$GDD(n, 2, 6; 5n, 5/(2(n-1)))$
$(n, 5, 10)$	$n \equiv 1 \pmod{2}$	$GDD(n, 2, 6; 10n, 5(n-1))$
$(n, 5, 20)$	All n	$GDD(n, 2, 6; 20n, 10(n-1))$

TABLE 6. Summary of Minimal Indices for Configuration (1, 5)

n	λ_1	λ_2
$n \equiv 0, 6, 10, 11, 15, 16 \pmod{20}$	$2n$	$(n-1)$
$n \equiv 1, 5 \pmod{20}$	n	$(n-1)/2$
$n \equiv 2, 4, 8, 12, 14, 18 \pmod{20}$	$10n$	$5(n-1)$
$n \equiv 3, 7, 9, 13, 17, 19 \pmod{20}$	$5n$	$5(n-1)/2$

However, this construction does not always give optimal values of λ_1 and λ_2 . By Theorem 1.5, we have the following relation between λ_1 and λ_2 .

Corollary 6.2. *For any configuration (1, 5) $GDD(n, 2, 6; \lambda_1, \lambda_2)$ we have $\lambda_2 = \frac{\lambda_1(n-1)}{2n}$.*

From Theorem 6.2 we see that for some value of n the minimum possible indices are (λ_1, λ_2) . As in the other two configurations, we may find the minimal indices by Theorem 6.2 and Theorem 1.1. Further, any other GDD with configuration (1, 5) will have indices $(w\lambda_1, w\lambda_2)$ for some positive integer w . The minimal indices are summarized in the next theorem.

Theorem 6.3. *The minimal indices (λ_1, λ_2) for any configuration (1, 5) $GDD(n, 2, 6; \lambda_1, \lambda_2)$ summarized in Table 6.*

Proof. By Theorem 6.2, we have that $\lambda_2 = \frac{\lambda_1(n-1)}{2n}$. This implies that if $n \equiv 1 \pmod{2}$ then λ_1 must be a multiple of n and λ_2 must be a multiple of $(n-1)/2$. However, if $n \equiv 11, 15 \pmod{20}$ then the indices given do not give an even number of blocks which is required by Theorem 1.4. So for $n \equiv 11, 15 \pmod{20}$, if we take two times the minimum possible indices, the number of blocks will be integer valued implying $(\lambda_1, \lambda_2) = (2n, (n-1))$. Also, using the given indices for $n \equiv 3, 7, 9 \pmod{10}$ results in a non-integer value for the number of blocks given by Theorem 1.1. Thus we must take 5 times these, so the minimal indices are $(\lambda_1, \lambda_2) = (5n, 5(n-1)/2)$. Finally, if $n \equiv 1, 5 \pmod{20}$, the necessary conditions in Theorem 1.1 are met.

If $n \equiv 0 \pmod{2}$, Theorem 6.2 tells us that λ_1 must be a multiple of $2n$ and λ_2 must be a multiple of $n-1$. However if $n \equiv 2, 4, 8 \pmod{10}$, then these values give a non-integer value for the number of blocks. If we take 5 times these indices then the necessary condition in Theorem 1.1 is satisfied, and so the minimal indices are $(\lambda_1, \lambda_2) = (10n, 5(n-1))$. Notice that for $n \equiv 0, 6 \pmod{10}$, the given indices are $(\lambda_1, \lambda_2) = (2n, n-1)$ which are the minimum possible. \square

7. CONSTRUCTING CONFIGURATION (1,5) GDDs

In this section we focus on constructing (1, 5) GDDs with minimal indices. Theorem 6.1 gives us the following results.

Corollary 7.1. *The necessary conditions are sufficient for the existence of a configuration (1, 5) $GDD(n, 2, 6; n, (n-1)/2)$ for $n \equiv 1, 5 \pmod{20}$.*

Corollary 7.2. *The necessary conditions are sufficient for the existence of a configuration (1, 5) $GDD(n, 2, 6; 2n, n-1)$ for $n \equiv 11, 15 \pmod{20}, n \neq 15$.*

Notice that in the previous two constructions, the design is minimal. We use a resolvable BIBD($n, 5, 4$) in the following construction. In [1], it is given that a resolvable BIBD($n, 5, 4$) exists for $n \equiv 0 \pmod{10}$ except for $n = 10, 160, 190$.

Theorem 7.3. *Let $n \equiv 0 \pmod{10}, n \neq 10, 160, 190$. Then the necessary conditions are sufficient for the existence of a configuration $(1, 5) \text{ GDD}(n, 2, 6; 2n, n - 1)$.*

Proof. Let $n \equiv 0 \pmod{10}, n \neq 10, 160, 190$. Assume the two groups are $A = \{1, 2, \dots, n\}$ and $A' = \{1', 2', \dots, n'\}$. There exists a RBIBD($n, 5, 4$) with $b = n(n - 1)/5$ blocks, and each point appearing $r = (n - 1)$ times. Let D be such a design on A with parallel classes C_1, C_2, \dots, C_{n-1} . Construct a graph G in the following manner. For $j = 1, 2, \dots, n - 1$, create the bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of C_j and $V(G_j^2)$ are the points in A' . Each of the first $n/10$ vertices in G_j^1 are adjacent to the vertices in G_j^2 that correspond to the first $n/10$ blocks of C_j . Each of the last $n/10$ vertices in G_j^2 are adjacent to the vertices in G_j^2 that correspond to the last $n/10$ blocks of C_j . Thus each vertex in G_j^1 has degree $n/2$ and each vertex in G_j^2 has degree $n/10$. This creates half of the desired blocks in the GDD. To obtain the other half, let D be an RBIBD($n, 5, 4$) on A' and repeat the construction. This time, each of the first $n/10$ vertices in G_j' will be adjacent to the vertices in G_j^2 that correspond to the last $n/10$ blocks of C_j , and each of the last $n/10$ vertices of G_j^1 will be adjacent to the vertices in G_j^2 that correspond to the first $n/10$ blocks of C_j .

In the design, each pair appears four times and will be matched $n/2$ times. Now consider a second pair of associates $\{x, y'\}$ where $x \in A$ and $y' \in A'$. The points x and y appear in every parallel class exactly once. So for each G_j , if x and y are both in the same half of A (either in the first $n/10$ blocks of C_j or the last $n/10$ blocks of C_j) then $\{x, y'\}$ appears once in the first part of the construction and zero times in the second part. If x and y were in different halves of A , then $\{x, y'\}$ appears once in the second part of the construction and zero times in the first part. Therefore $\{x, y'\}$ appears exactly once per G_j . Thus λ_2 is the number of parallel classes or $n - 1$. \square

A near parallel class is a partial parallel class missing a single point. A near -resolvable design NRB($n, k, k - 1$) is a BIBD($n, k, k - 1$) with the property that the blocks can be partitioned into near parallel classes. For such a design, every point is absent from exactly one class. The necessary condition for the existence of an NRB($v, k, k - 1$) is $v \equiv 1 \pmod{k}$. It is known that the necessary condition is sufficient for the existence of a NRB($v, k, k - 1$) if $k \leq 7$ (see [1]). We use near resolvable designs in the following construction.

Theorem 7.4. *Let $n \equiv 6 \pmod{10}$. Then the necessary conditions are sufficient for the existence of a configuration $(1, 5) \text{ GDD}(n, 2, 6; 2n, n - 1)$.*

Proof. Let $n \equiv 6 \pmod{10}$, and the two groups have point sets $A = \{1, 2, \dots, n\}$ and $A' = \{1', 2', \dots, n'\}$. Since $n \equiv 6 \pmod{10}$, there exists a NRB($n, 5, 4$). It has n near parallel classes with $(n - 1)/5$ blocks in them each. Let D be such a design on the point set of A , and resolve the blocks of D into near parallel classes C_1, C_2, \dots, C_n where C_i misses point i . Construct a graph G in the following manner. For $j = 1, 2, \dots, n/2$, create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of C_j and $V(G_j^2)$ are the points $\{1', 2', \dots, n/2'\}$. For

$j = n/2 + 1, \dots, n$, create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^2)$ are the points $\{(n/2 + 1)', (n/2 + 2)', \dots, n'\}$. This creates half of the desired blocks. To get the other half, let D be the NRB($n, 5, 4$) on A' and repeat the construction with $V(G_j^2)$ being the points $\{1, 2, \dots, n/2\}$ for $j = n/2 + 1, \dots, n$, and $V(G_j^2)$ being the points $\{(n/2) + 1, \dots, n\}$ for $j = 1, 2, \dots, n/2$.

Consider a pair of first associates. It will appear $4(n/2) = 2n$ times in a block of size 6. Now consider a pair of second associates where $x \in A$ and $y' \in A'$. If $x \in \{1, 2, \dots, n/2\}$ and $y' \in \{1', 2', \dots, (n/2)'\}$ then $\{x, y'\}$ will appear $(n/2) - 1$ times in the first part of the construction and $n/2$ times in the second. It is the same case if $x \in \{n/2 + 1, n/2 + 2, \dots, n\}$ and $y' \in \{(n/2 + 1)', (n/2 + 2)', \dots, n'\}$. If $x \in \{1, 2, \dots, n/2\}$ and $y' \in \{(n/2 + 1)', (n/2 + 2)', \dots, n'\}$, then $\{x, y'\}$ will appear $n/2$ times in the first part and $n/2 - 1$ times in the second part. It is the same case if $x \in \{n/2 + 1, n/2 + 2, \dots, n\}$ and $y' \in \{1', 2', \dots, n'\}$. Thus $\lambda_2 = n - 1$. \square

Note that we have constructed minimal GDDs for $n \equiv 0, 1, 5, 6 \pmod{10}$ (for all but a few values). Recall that a near-minimal design is one that has exactly twice the minimal indices. By Theorem 6.1, the necessary conditions are sufficient for the existence of a near minimal GDD($n, 2, 6; \lambda_1, \lambda_2$) for $n \equiv 2, 3, 4, 7, 8, 9 \pmod{10}$. We may construct a minimal GDD($n, 2, 6; \lambda_1, \lambda_2$) for $n \equiv 3, 7, 9 \pmod{10}$ given the existence of a 5-resolvable BIBD($n, 5, 10$).

Theorem 7.5. *The existence of a 5-resolvable BIBD($n, 5, 10$) implies the existence of a configuration $(1, 5)$ GDD($n, 2, 6; 5n, 5(n - 1)/2$) for $n \equiv 3, 7, 9 \pmod{10}$.*

Proof. Let $n \equiv 3, 7, 9 \pmod{10}$ and assume there exists a 5-resolvable BIBD($n, 5, 10$). Assume the two groups are $A = \{1, 2, 3, \dots, n\}$ and $A' = \{1', 2', 3', \dots, n'\}$ and let D be such a design on point set A . Resolve the blocks of D into 5-parallel classes $C_1, C_2, \dots, C_{n-1/2}$, each having n blocks. Construct a graph G in the following manner. For $j = 1, 2, \dots, (n - 1)/4$, create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of C_j and $V(G_j^2)$ are the odd numbers in A' . For $j = (n - 1)/4 + 1, \dots, (n - 1)/2$, create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of C_j and $V(G_j^2)$ are the even numbers in A' . This creates half of the desired blocks. To get the other half, let D be a 5-RBIBD($n, 5, 10$) on A' and repeat the construction with $V(G_j^2)$ being the even numbers in A for $j = 1, 2, \dots, (n - 1)/4$ and $V(G_j^2)$ being the odd numbers in A for $j = (n - 1)/4 + 1, \dots, (n - 1)/2$.

Consider a pair of first associates. It will appear 10 times in D . Therefore, in the given construction it will appear $5n$ times in a block of size 6. Now consider a pair of second associates $\{x, y'\}$. In each part of the construction, this pair appears $5(n - 1)/4$ times, thus it appears a total of $5(n - 1)/2$ times. \square

7.1. Summary of Minimality.

TABLE 7. Summary of Constructions and Minimality for Configuration $(1, 5)$

n	λ_1	λ_2	
$n \equiv 0, 10, 11, 15, 6, 16 \pmod{20}, n \neq 10, 15, 160, 190$	$2n$	$(n - 1)$	minimal
$n \equiv 1, 5 \pmod{20}$	n	$(n - 1)/2$	minimal
$n \equiv 2, 4, 8, 12, 14, 18 \pmod{20}$	$10n$	$5(n - 1)$	near-minimal
$n \equiv 3, 7, 9, 13, 17, 19 \pmod{20}$	$5n$	$5(n - 1)/2$	near-minimal

We conclude this section with a summary of the GDDs we have constructed, and their minimality found in Table 7.

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